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Consistent construction of perturbation theory on non-commutative spaces

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Abstract. We examine the effect of non-local deformations on the applicability of interaction point time ordered perturbation theory (IPTOPT) based on the free Hamiltonian of local theories. The usual argument for the case of quantum field theory on a non-commutative space (based on the fact that the introduction of star products in bilinear terms does not alter the action) is not applicable to IPTOPT due to several discrepancies compared to the naive path integral approach when non-commutativity involves time. These discrepancies are explained in detail. Besides scalar models, gauge fields are also studied. For both cases, we discuss the free Hamiltonian with respect to non-local deformations.

1 Introduction

It is widely believed that the usual concept of space-time locally modelled as flat Minkowski space breaks down at distances of the magnitude of the Planck scale. One attempt to describe physics at such small scales is to replace the commutative space-time coordinates x^μ by non-commutative (NC) operators $\hat{x}^\mu$ implying uncertainty relations among the coordinates [1]. The simplest model one can study is characterised by the following commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{1}$$

with $\theta^{\mu\nu}$ representing a real, constant, antisymmetric tensor. We will study this model realised by the so-called (Moyal–Weyl) star product replacing ordinary local products of fields within the usual quantum field theory (QFT), referred to as NCQFT later on. This star product is defined for real-analytic, L^1 functions f, g as

$$(f * g)(x) \equiv \mathrm{e}^{\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\mu^\zeta \partial_\nu^\eta} f(x+\zeta) g(x+\eta)\Big|_{\zeta=\eta=0}. \tag{2}$$

It is important to note the infinitely many derivatives acting within this product. Concerning QFT, especially the time derivatives present for $\theta^{0i} \neq 0$ turn out to be problematic. In that case, a violation of unitarity has been observed [2] when applying the rules given in [3]. Unitarity could be reestablished in [4] using the Yang–Feldmann equation [5] and in [1,6,7] by applying IPTOPT. Below, we will make clear that this version of time ordering is a consequence of quantum mechanical basics. IPTOPT was worked out in a more general context [8] applicable to a large class of non-local interactions of scalar particles. The non-locality can be realised by the integral representation of the star product [9] as

$$(f \star g)(x) = \int \mathrm{d}^4 s \int \frac{\mathrm{d}^4 l}{(2\pi)^4}\, f\left(x - \frac{1}{2}\tilde{l}\right)\, g(x+s)\, \mathrm{e}^{\mathrm{i}ls},$$
$$\tilde{l}^\nu := l_\mu \theta^{\mu\nu}. \tag{3}$$

This makes the effect of time ordering more transparent. The problems occurring for $\theta^{0i} \neq 0$ can be identified with the non-locality in time. Besides unitarity, the finite UV/IR mixing behaviour [10] is a further advantage of IPTOPT. At this point, we want to mention that in θ-expanded field theories [11–13] these difficulties do not appear. However, they might not be renormalisable [14,15].

So long, deformed field theory has been pursued in a somewhat ambivalent way: The Moyal product has been used in the interaction part of the Lagrangian, the bilinear term remained unchanged [3] due to the argument that in the action one star product can always be omitted:

$$\int \mathrm{d}^4 x\, (f \star g)(x) = \int \mathrm{d}^4 x\, f(x) g(x). \tag{4}$$

However, this is not directly applicable to the approach based on IPTOPT. This is indicated by the differences between the Feynman rules given in [3] and the complicated ones based on IPTOPT [8]. There it was realised that quadratic parts in the interaction do have a θ-dependent contribution for $\theta^{0i} \neq 0$. This raises the question whether the free Hamiltonian in the framework of IPTOPT gets deformed or not when introducing the star product (3) and

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i.g. non-local deformations. We will generalise the results of [1] to the case of $U(1)$ gauge fields, and allow for more general non-local deformations. We will also construct IPTOPT beginning with the Schrödinger equation to make the definition of time ordering clear. The discrepancies between IPTOPT and a naive path integral approach giving the simple Feynman rules [3] will be explained.

In Sect. 5, we will discuss the canonical deformation of gauge field theory and BRST-symmetry also in the Hamiltonian approach.

2 Quantum mechanics

In this section, quantum mechanical basics needed later are reviewed. After the introduction of various types of time dependence of operators and states time ordered perturbation theory is discussed.

2.1 Schrödinger picture

We start with the Schrödinger equation (with $\hbar = 1$),

$$\mathrm{i}\frac{\partial}{\partial t}|Z_{\mathrm{S}}(t)\rangle = H_{\mathrm{S}}|Z_{\mathrm{S}}(t)\rangle, \tag{5}$$

where H_{S} is the Hamilton operator and $|Z_{\mathrm{S}}(t)\rangle$ is a time dependent Schrödinger state. As long as H_{S} is time independent, we have the simple solution

$$|Z_{\mathrm{S}}(t)\rangle = \mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}(t-t_0)}|Z_{\mathrm{S}}(t_0)\rangle, \tag{6}$$

with some initial state $|Z_{\mathrm{S}}(t_0)\rangle$.

Now the question is: What is the particular feature of a specific model described by (5)? The answer is simple: Different models are distinguished by different Hamiltonians. In particular, the states are defined as solutions of (5) with a particular Hamiltonian H_{S}.

In the Schrödinger picture (which is denoted by the index S) we have the notion of a time independent Hamiltonian which generates time dependent states. Physics is described via matrix elements of operators with these states. Those operators are assumed to be time independent and (if we are lucky) known, so the interesting thing is to get the correct states.

2.2 Heisenberg picture

But if we have a closer look at the matrix elements of some time independent operator A_{S},

$$\begin{aligned}\langle A\rangle &= \langle Z_{\mathrm{S}}(t)|A_{\mathrm{S}}|Z_{\mathrm{S}}(t)\rangle \\ &= \langle Z_{\mathrm{S}}(t_0)|\mathrm{e}^{+\mathrm{i}H_{\mathrm{S}}(t-t_0)}A_{\mathrm{S}}\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}(t-t_0)}|Z_{\mathrm{S}}(t_0)\rangle,\end{aligned} \tag{7}$$

we could also argue that we have time independent states $|Z_{\mathrm{H}}\rangle := |Z_{\mathrm{S}}(t_0)\rangle$ and time dependent *Heisenberg* operators A_{H} (let $t_0 = 0$),

$$A_{\mathrm{H}}(t) := \mathrm{e}^{+\mathrm{i}H_{\mathrm{S}}t}A_{\mathrm{S}}\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t}. \tag{8}$$

This is the Heisenberg picture. Instead of fixing the operators and searching for time dependent states, we keep the states fixed and put our interest in time evoluting operators.

Instead of the Schrödinger equation (5) for the states, we now have the Heisenberg equation for the Heisenberg operators, obtained from differentiating (8) with respect to the time (note that A_{S} does not explicitly depend on time),

$$-\mathrm{i}\frac{\partial}{\partial t}A_{\mathrm{H}}(t) = [H_{\mathrm{H}}, A_{\mathrm{H}}(t)]. \tag{9}$$

Here, $H_{\mathrm{H}} = H_{\mathrm{S}}$ is still time independent. The Heisenberg equation looks indeed very similar to the Schrödinger equation.

2.3 Dirac picture

Somehow in between is the Dirac picture, where states and operators have a time evolution. The *free* part $H_{0\mathrm{S}}$ of the Hamiltonian $H_{\mathrm{S}} = H_{0\mathrm{S}} + H_{\mathrm{IS}}$ is used to describe the time evolution of the operators, whereas the *interaction part* H_{IS} will describe the time evolution of the states. The states in the interaction picture are defined as

$$|Z_{\mathrm{D}}(t)\rangle := \mathrm{e}^{\mathrm{i}H_{0\mathrm{S}}t}|Z_{\mathrm{S}}(t)\rangle\,. \tag{10}$$

From $\langle Z_{\mathrm{S}}|A_{\mathrm{S}}|Z_{\mathrm{S}}\rangle = \langle Z_{\mathrm{D}}|A_{\mathrm{D}}|Z_{\mathrm{D}}\rangle$ we conclude

$$A_{\mathrm{D}}(t) := \mathrm{e}^{+\mathrm{i}H_{0\mathrm{S}}t}A_{\mathrm{S}}\mathrm{e}^{-\mathrm{i}H_{0\mathrm{S}}t}\,. \tag{11}$$

With

$$\begin{aligned}&H_{0\mathrm{D}} = H_{0\mathrm{S}}, \quad H_{\mathrm{ID}}(t) := \mathrm{e}^{+\mathrm{i}H_{0\mathrm{S}}t}H_{\mathrm{IS}}\mathrm{e}^{-\mathrm{i}H_{0\mathrm{S}}t}, \\ &[H_{0\mathrm{S}}, \exp(\pm\mathrm{i}H_{0\mathrm{S}}t)] = 0, \\ &H_{\mathrm{ID}}(t=0) = H_{\mathrm{IS}},\end{aligned} \tag{12}$$

we find the two evolution equations (from (5) and (10), respectively)

$$\begin{aligned}-\mathrm{i}\frac{\partial}{\partial t}A_{\mathrm{D}}(t) &= [H_{0\mathrm{D}}, A_{\mathrm{D}}(t)], \\ \mathrm{i}\frac{\partial}{\partial t}|Z_{\mathrm{D}}(t)\rangle &= H_{\mathrm{ID}}(t)|Z_{\mathrm{D}}(t)\rangle.\end{aligned} \tag{13}$$

We see that the time evolution of the operators is defined by the *free* Hamiltonian, so that $A_{\mathrm{D}}(t)$ is simply a solution of the free theory. The time evolution of the states, on the other hand, depends only on the *interaction* Hamiltonian. Note that, since the free Hamiltonians in the Schrödinger and Dirac picture are the same, we define $H_{0\mathrm{S}} = H_{0\mathrm{D}} \equiv H_0$ from now on.

2.4 IPTOPT

Given some interaction Hamiltonian $H_{\mathrm{ID}}(t)$ in the Dirac picture, the evolution equation (13) for the Dirac states

can be solved by introducing the time evolution operator $U(t,t_0)$ as

$$U(t,t_0)|Z_{\rm D}(t_0)\rangle \equiv |Z_{\rm D}(t)\rangle. \tag{14}$$

The well known solution of the resulting differential equation with the boundary condition $U(t_0,t_0)=1$ is

$$\begin{aligned} &U(t,t_0)\\ &=\sum_{n=0}^{\infty}\frac{(-\mathrm{i})^n}{n!}\int_{t_0}^{t}\mathrm{d}t_1\ldots\mathrm{d}t_n\,T\{H_{\rm ID}(t_1)\ldots H_{\rm ID}(t_n)\}\\ &\equiv T\mathrm{e}^{-\mathrm{i}\int_{t_0}^{t}H_{\rm ID}(t')\mathrm{d}t'}. \end{aligned} \tag{15}$$

The time ordering operator is denoted by T, and it should be pointed out here that it rearranges the whole operators $H_{\rm ID}(t)$ according to their time argument t. It does not act on parts of $H_{\rm ID}(t)$. We call this *interaction point time ordering* (IPTO) to distinguish it from other possible time orderings of objects. Since we are considering non-local deformations of the classical theory, the concept of interaction at a point is no longer valid. By *interaction point* we mean the time argument of the interaction Hamiltonian in (15). The interaction Hamiltonian may of course be a non-local expression and contain products of fields at different times; cf. (3).

In order to describe scattering processes, we need the S-operator defined by

$$S\equiv U(\infty,-\infty)=T\mathrm{e}^{-\mathrm{i}\int\mathrm{d}t\,H_{\rm ID}(t)}. \tag{16}$$

Again, T acts on the time argument t of $H_{\rm ID}(t)$. Only for simple theories such as scalar ϕ^4-theory, one can write $H_{\rm ID}(t)=-\int\mathrm{d}^3x\,\mathcal{L}_{\rm ID}(t,\boldsymbol{x})$. But note that IPTOPT principally requires the interaction Hamiltonian and not the Lagrangian.

S-matrix elements are thus given by

$$S_{fi}:=\langle f|S|i\rangle, \tag{17}$$

where $|i\rangle$ and $\langle f|$ represent the incoming and outgoing states, respectively.

3 Quantum field theory

The operators we have dealt with in the last section can usually be expressed in terms of field operators ϕ and their canonical conjugates π. Their operator character is manifested by the equal time commutation relations

$$\begin{aligned} &[\pi(t,\boldsymbol{x}),\pi(t,\boldsymbol{x}')]=[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{x}')]=0,\\ &[\pi(t,\boldsymbol{x}),\phi(t,\boldsymbol{x}')]=-\mathrm{i}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \end{aligned} \tag{18}$$

For the application of IPTOPT the free field operators, which are given in the Dirac picure, are especially important. Their dynamics can be characterised by the Lagrange or Hamilton formalism, as we will briefly discuss in this section. The main investigation of this section is the effect of deformation on the free Hamiltonian $H_0\to H_0^*$. Star products in the action can be left out for quadratic terms, which is often given as the reason that the free theory on non-commutative spaces is the same as for the commutative one. In the Hamiltonian, however, the star product does not automatically (only for $\theta^{0i}=0$) drop out of quadratic terms in the interaction. This is the reason why we address the deformation of the free Hamiltonian from another point of view.

3.1 Commutative space

The free scalar field is described by the equation of motion

$$(\Box+m^2)\phi=0. \tag{19}$$

This equation of motion can be obtained by a field variation of the action,

$$\begin{aligned} W&=\int\mathrm{d}tL_0=\int\mathrm{d}^4x\mathcal{L}_0,\\ \mathcal{L}_0&=\frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi)-\frac{1}{2}m^2\phi^2,\\ \delta W=0&\Rightarrow(\Box+m^2)\phi=0. \end{aligned} \tag{20}$$

Another possibility is the description via the Hamiltonian $H=\int\mathrm{d}^3x\mathcal{H}$,

$$\pi:=\frac{\partial\mathcal{L}}{\partial\dot\phi}=\dot\phi,\quad \mathcal{H}:=\dot\phi\pi-\mathcal{L}. \tag{21}$$

An explicit calculation yields

$$H_0=\int\mathrm{d}^3x\frac{1}{2}(\dot\phi^2+(\boldsymbol{\nabla}\phi)^2+m^2\phi^2)\geq 0. \tag{22}$$

H_0 is interpreted as the total energy of the system. Energy conservation $\frac{\mathrm{d}}{\mathrm{d}t}H_0=\frac{\mathrm{d}}{\mathrm{d}t}\int\mathrm{d}^3x\,\mathcal{H}_0=0$ is obtained by use of partial integration and the equation of motion.

Note that ϕ and π correspond to x (the current elongation) and p (the momentum) of the harmonic oscillator, whereas the space coordinates $\boldsymbol{x}$ could be thought of as "labels" of the infinitely many harmonic oscillators hanging around in space. Only time is always time.

Since ϕ satisfies the free field equation, we can write it as $\phi(x)=\phi^+(x)+\phi^-(x)$, where

$$\begin{aligned} \phi^-(x)&=\frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\,a^-(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}x_\mu k^{+\mu}},\\ \phi^+(x)&=\frac{1}{(2\pi)^{3/2}}\int\frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\,a^+(\boldsymbol{k})\,\mathrm{e}^{+\mathrm{i}x_\mu k^{+\mu}}. \end{aligned} \tag{23}$$

Here we have $k^+=(\omega_k,\boldsymbol{k})$, $\omega_k=\sqrt{\boldsymbol{k}^2+m^2}$. The combination with (18) yields

$$[a^-(\boldsymbol{k}),a^+(\boldsymbol{k}')]=\delta^3(\boldsymbol{k}-\boldsymbol{k}'). \tag{24}$$

Finally the Hamiltonian can be rewritten as

$$H_0=\int\mathrm{d}^3k\,\omega_k\,\frac{1}{2}(a^+(\boldsymbol{k})a^-(\boldsymbol{k})+a^-(\boldsymbol{k})a^+(\boldsymbol{k})). \tag{25}$$

3.2 Non-local deformation

In this subsection, we want to study the effect of general non-local deformations on the free scalar Hamiltonian. At the end, we will examine some examples, such as canonical deformation. Let us start with the usual Hamiltonian

$$H_0 = \int \mathrm{d}^3x \frac{1}{2}(\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2). \tag{26}$$

Now, let us introduce a non-local deformation of the above pointwise product:

$$f(x)g(x) \to \int \mathrm{d}\underline{\mu}\, w(\underline{\mu})\, f(x + h_1(\underline{\mu}))\, g(x + h_2(\underline{\mu})). \tag{27}$$

An n-tuple of real numbers is denoted by $\underline{\mu}$, and the h's represent the non-localities mapping $\underline{\mu}$ into a four-vector (see [8] for details).[1] Furthermore, note that not all choices of w, h_1, and h_2 make sense. For example, it will become clear below that certain choices are not even lower bounded. For H_0 one thus gets

$$\begin{aligned} H_0(t) &\to H_0^*(t) \\ &= \frac{1}{2}\int \mathrm{d}^3x \int \mathrm{d}\underline{\mu}\, w(\underline{\mu}) \\ &\quad \times\big(\partial^\nu\phi(x + h_1(\underline{\mu}))\partial^\nu\phi(x + h_2(\underline{\mu})) \\ &\quad +m^2\phi(x + h_1(\underline{\mu}))\phi(x + h_2(\underline{\mu}))\big). \end{aligned} \tag{28}$$

Still, $\phi(x)$ shall denote the free field operator obeying the usual free field equation with physical mass m given by

$$\partial^\mu\partial_\mu\phi - m^2\phi = 0. \tag{29}$$

Therefore, we can apply Fourier transformation and interpret the coefficients as creation and annihilation operators $a^\dagger(\mathbf{k})$ and $a(\mathbf{k})$, respectively. Straightforward calculation yields

$$\begin{aligned} H_0^*(t) &= \frac{1}{2}\int \mathrm{d}^3k \int \mathrm{d}\underline{\mu}\, w(\underline{\mu}) \\ &\quad \times\big(a(\mathbf{k})a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}k^+(h_1(\underline{\mu})-h_2(\underline{\mu}))} \\ &\quad +a^\dagger(\mathbf{k})a(\mathbf{k})\mathrm{e}^{-\mathrm{i}k^+(h_1(\underline{\mu})-h_2(\underline{\mu}))}\big). \end{aligned} \tag{30}$$

The coefficients of the terms proportional to $a^\dagger(\mathbf{p})a^\dagger(\mathbf{k})$ and $a(\mathbf{p})a(\mathbf{k})$ vanish:

$$(p^{+\mu}k^{+\mu} + m^2)\delta^3(\mathbf{k} + \mathbf{p}) = 0,$$

whereas the coefficient of $a(\mathbf{p})a^\dagger(\mathbf{k})$ and $a^\dagger(\mathbf{p})a(\mathbf{k})$ is proportional to

$$(-p^{+\mu}k^{+\mu} + m^2)\delta^3(\mathbf{k} + \mathbf{p}) = 2\omega_p^2\delta^3(\mathbf{p} + \mathbf{k}).$$

[1] At this point, this notation might look unnecessarily complicated, and one might ask, why not use two four-vectors a_1 and a_2 instead of the non-localities $h_1(\underline{\mu})$ and $h_2(\underline{\mu})$. However, we prefer to be consistent with the notation of [8]. Also, the notation used here seems to be more general.

Usually, the Hamiltonian is normal ordered. For H_0^* we obtain

$$:H_0^*(t) := \int \mathrm{d}^3k\, \omega_k\, a^\dagger(\mathbf{k})a(\ k)\xi(\ k), \tag{31}$$

where ξ is given by

$$\xi(\mathbf{k}) = \int \mathrm{d}\underline{\mu}\, w(\underline{\mu})\, \cos(k^+(h_1(\underline{\mu}) - h_2(\underline{\mu}))). \tag{32}$$

Note that for certain choices of w, h_1, and h_2, $\xi(\mathbf{k})$ and consequently also H_0 might not be lower bounded. If $\xi(\mathbf{k})$ is constant, H_0^* and H_0 only differ by an overall normalisation constant. Therefore, the free Hamiltonian would be unaltered. But what deformations do that job?

A trivial solution to this question is the choice $h_1(\underline{\mu}) = h_2(\underline{\mu})$ and the requirement $\int \mathrm{d}(\underline{\mu})\, w(\underline{\mu}) < \infty$. Actually this deformation is still local.

The next example is canonical deformation, discussed in the Introduction. We have $\underline{\mu} = \{l, s\}$, $w(\underline{\mu}) = \exp(\mathrm{i}sl)/(2\pi)^4$, $h_1(\underline{\mu}) = -\frac{1}{2}\tilde{l}$ and $h_2(\underline{\mu}) = s$. Thus, we obtain for ξ

$$\xi(\mathbf{k}) = \mathrm{e}^{-\mathrm{i}k^+\tilde{k}^+/2} = 1, \tag{33}$$

and the free Hamiltonian is unaltered by the deformation [1]. The use of the perturbation theory worked out in [6, 8, 9] is justified.

As a third example, we consider the approach to UV-finite theories considered in [16]. Comparison with our definitions yields

$$\begin{aligned} &\underline{\mu} = \{a_1, a_2\},\ h_1(\underline{\mu}) = \zeta a_1,\ h_2(\underline{\mu}) = \zeta a_2, \\ &w(\underline{\mu}) = 2c_2\mathrm{e}^{-\frac{1}{2}(a_1^{\nu 2} + a_2^{\nu 2})}\delta^4(a_1 + a_2). \end{aligned}$$

Therefore, we have

$$\xi(\mathbf{k}) = 2c_2\pi^2\mathrm{e}^{-\zeta^2 k_\nu^{+2}}, \tag{34}$$

where ζ has the dimension of length in order to keep the parameters a_i dimensionless and to provide control over the non-locality, and c_2 represents a normalization constant. With a suitable choice for c_2 we get the following Hamiltonian:

$$:H_0^*(t) := \int \mathrm{d}^3k\, \omega_k'\, a^\dagger(\mathbf{k})a(\ k)\xi(\ k), \tag{35}$$

where

$$\omega_k' = \omega_k\mathrm{e}^{-\zeta^2 k_\nu^{+2}}. \tag{36}$$

In principle, this Hamiltonian can be interpreted as the new free particle energy.

The simplified UV-finite QFT introduced in [17] (cf. also [18]) gives a similar result. We have $\underline{\mu} = \{b_1, b_2\}$ (b_1 and b_2 are 4-dimensional vectors), $w(\underline{\mu}) = \exp(-{b_1}^{\mathrm{T}}b_1 - {b_2}^{\mathrm{T}}b_2)$, $h_1(\underline{\mu}) = M\, b_1$ and $h_2(\underline{\mu}) = M\, b_2$. Hence, ξ is given by

$$\xi(\mathbf{k}) = \mathrm{e}^{-\frac{1}{2}k^{+\mathrm{T}}\kappa\, k}. \tag{37}$$

In the last two examples the deformed free Hamiltonian does not equal the undeformed one. Also, the interpretation of ω_k' as the energy of a physical state with momentum $\mathbf{k}$ is troublesome, since the energy goes to zero for large $\mathbf{k}$. Therefore, the deformation has only been introduced in the interaction terms in [16, 17], respectively.

4 A conceptual note

In this section, we want to discuss discrepancies between IPTOPT and the naive path integral approach (NPIA). By IPTOPT we mean calculations according to (15) with

$$H_{\mathrm{ID}}(x^0) \equiv \frac{\lambda}{k!}\int \mathrm{d}^3x\ (\phi*)^k(x), \tag{38}$$

and ϕ denoting the field operators in the Dirac picture. The resulting Feynman rules are given in [8]. By NPIA we mean that one calculates n-point functions according to the path integral

$$\int \mathcal{D}\phi\ \phi(x_1)\dots\phi(x_n)\ \mathrm{e}^{\mathrm{i}I[\phi]}, \tag{39}$$

with I denoting the corresponding action including $\mathrm{i}\epsilon$ terms:

$$I[\phi] = \int \mathrm{d}^4x\ \left[\mathcal{L}_0 + \frac{\lambda}{k!}(\phi*)^k(x) + \mathrm{i}\epsilon\right]. \tag{40}$$

The corresponding Feynman rules are the same as for the local theory but with phase factors to be included for vertices [3]. Meanwhile, it is clear that these two approaches differ when non-commutativity involves time. The most striking problem is the unitarity violation [2] when applying these naive Feynman rules, which can be cured by a strict application of IPTOPT [4,7]. Another mismatch was realised in [8], where it turned out that the star product of NCQFT does not drop out of some quadratic terms which might be considered as counterterms for example. But in the Lagrangian (NPIA), star products become redundant in any bilinear term.

For the local version of the model we are studying, both, the NPIA and IPTOPT, give the same results. But for non-local theories as NCQFT, these approaches are not equivalent. To see the crucial points about these mismatches, we sketch how to pass from IPTOPT to the NPIA [19]. One starts with the Hamiltonian $H_{\mathrm{S}} \equiv H_{\mathrm{D}}(0)$ written as a functional of canonical field operators $\phi_{\mathrm{D}}(t,\boldsymbol{x}), \pi_{\mathrm{D}}(t,\boldsymbol{x})$ in the Dirac picture

$$H_{\mathrm{D}}(t) \equiv H[\phi_{\mathrm{D}}, \pi_{\mathrm{D}}; t]. \tag{41}$$

To be specific, our version of IPTOPT and NCQFT gives

$$H_{\mathrm{D}}(t) = H_0 + H_{\mathrm{ID}}(t) \tag{42}$$

combined with (38). The special notation of the functional (41) is due to the non-localities, especially the ones in time: it is not a functional depending just on field operators given at fixed time t, but all possible times are involved (see also (52) below). Furthermore, one assumes two complete basis sets $|q;t\rangle$ and $|p;t\rangle$ for each time t being eigenstates of the field operators in the Heisenberg picture $\phi_{\mathrm{H}}(t,\boldsymbol{x})$ and $\pi_{\mathrm{H}}(t,\boldsymbol{x})$, respectively. The goal is to evaluate scalar products $\langle q';t'|q;t\rangle$ between basis vectors given at different times $t<t'$. Then, one sandwiches sums over complete basis sets belonging to intermediate times t_i with $t<t_1\dots<t_N<t'$. So far, everything might also work for non-local field theories. Next, one has to evaluate matrix elements like

$$\langle q';t+\mathrm{d}t|q;t\rangle = \langle q';t|\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}\mathrm{d}t}|q;t\rangle. \tag{43}$$

At this point, H_{S} is usually rewritten as

$$\begin{aligned} H_{\mathrm{S}} &= \mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t}\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t}H[\phi_{\mathrm{D}},\pi_{\mathrm{D}};0] \\ &= \mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t}H[\phi_{\mathrm{D}},\pi_{\mathrm{D}};0]\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t}. \end{aligned} \tag{44}$$

In a local theory, the functional just depends on field operators $\phi_{\mathrm{D}}(0,\boldsymbol{x})$ evaluated at time $t=0$ which can be simply replaced using $\phi_{\mathrm{D}}(0,\boldsymbol{x}) = \phi_{\mathrm{H}}(0,\boldsymbol{x})$. As a matter of course, one thus rewrites this by sandwiching unit operators $\exp(\mathrm{i}H_{\mathrm{S}}t)\exp(-\mathrm{i}H_{\mathrm{S}}t)$ as

$$\begin{aligned} \mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t}H[\phi_{\mathrm{D}},\pi_{\mathrm{D}};0]\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t} &= \mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t}H[\phi_{\mathrm{H}},\pi_{\mathrm{H}};0]\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t} \\ &= H[\phi_{\mathrm{H}},\pi_{\mathrm{H}};t]. \end{aligned} \tag{45}$$

However, these steps are problematic for theories which are non-local in time. To see this, we consider an operator O_{D} defined as a Moyal product of two operators A_{D}, B_{D} in the Dirac picture:

$$\begin{aligned} O_{\mathrm{D}}(x) &\equiv (A_{\mathrm{D}} \star B_{\mathrm{D}})(x) \\ &= \int \mathrm{d}^4s \int \frac{\mathrm{d}^4l}{(2\pi)^4}\, e^{\mathrm{i}ls}\, A_{\mathrm{D}}\left(x-\frac{1}{2}\tilde{l}\right) B_{\mathrm{D}}(x+s). \end{aligned} \tag{46}$$

The subscript D at O_{D} indicating that O_{D} has the time dependence of an operator in the Dirac picture is justified since

$$O_{\mathrm{D}}(t,\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}H_0 t}O_{\mathrm{D}}(0,\boldsymbol{x})\mathrm{e}^{-\mathrm{i}H_0\ t}$$

holds. The transition from the Dirac to the Heisenberg picture is now done as usual [19]:

$$O_{\mathrm{H}}(t,\boldsymbol{x}) = \mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t}O_{\mathrm{D}}(0,\boldsymbol{x})\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t}. \tag{47}$$

Substituting (46), we get

$$\begin{aligned} O_{\mathrm{H}}(t,\boldsymbol{x}) &= \int \mathrm{d}^4s \int \frac{\mathrm{d}^4l}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}ls}\mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t} \\ &\quad \times A_{\mathrm{D}}\left(x_0-\frac{1}{2}\tilde{l}\right) B_{\mathrm{D}}(x_0+s)\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t} \\ &= \int \mathrm{d}^4s \int \frac{\mathrm{d}^4l}{(2\pi)^4} e^{\mathrm{i}ls} W(t,t-\tilde{l}^0/2)A_{\mathrm{H}}(x-\tilde{l}/2) \\ &\quad \times W(t-\tilde{l}^0/2,t+s^0)B_{\mathrm{H}}(x+s)W(t+s^0,t), \end{aligned} \tag{48}$$

where $x=(t,\boldsymbol{x})$, $x_0=(0,\boldsymbol{x})$, and

$$W(t,t_0) \equiv \mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t}\mathrm{e}^{-\mathrm{i}H_0(t-t_0)}\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t_0}. \tag{49}$$

$W(t,t_0)$ is unitary and $W(t,t)=1$, but in general $W(t,t_0)$ is not the unit operator. In order to stay consistent one thus has to redefine the non-commutative product with respect to Heisenberg fields correspondingly. For the Hamiltonian needed for path integrals, one could proceed with

$$\mathrm{e}^{\mathrm{i}H_{\mathrm{S}}t}H[\phi_{\mathrm{D}},\pi_{\mathrm{D}};0]\mathrm{e}^{-\mathrm{i}H_{\mathrm{S}}t} =: H'[\phi_{\mathrm{H}},\pi_{\mathrm{H}};t]. \tag{50}$$

When non-locality involves time clearly $H' \neq H$, and it can be expected that dealing with H' is pretty hard. We assume that $H' \neq H$ is the cause for the discrepancies between IPTOPT and the NPIA for systems described by Hamiltonians which are non-local in time. Further problems are expected when integrating out the conjugate momenta to pass from the Hamiltonian to the Lagrangian formulation, even if we accept H instead of H'.[2] This might be due to the fact that non-locality in time means that H depends on infinitely many time derivatives implying complicated equations of motion [20]. Furthermore, the equivalence of using the Lagrangian interaction or the Hamiltonian is not justified anymore by path integrals. We believe that it is important to check this explicitly. Besides derivative couplings and counterterms, NCQED might also be affected due to the complicated quantisation procedure which involves derivatives through the constraints [21].

Before dealing with gauge field models in the following section, we want to illustrate the differences of the NPIA and IPTOPT. The perturbation expansion of the NPIA is obtained by expanding the integrand in terms of the interaction leaving only the bilinear parts in the exponential. The resulting path integrals can be carried out and one gets Feynman rules which associate the usual propagators $\Delta_{\mathrm{F}}(x,y)$ (inverse of the bilinear parts) of the local theory with lines, and vertices contain four-momentum dependent phase factors. A time ordering interpretation of the resulting rules can be obtained by writing $\Delta_{\mathrm{F}}(x,y)$ in terms of time ordered products of the free annihilation and creation fields, ϕ^- and ϕ^+, respectively:

$$\begin{aligned}\Delta_{\mathrm{F}}(x,y) = {} & \tau(x^0-y^0)[\phi^-(x),\phi^+(y)] \\ & +\tau(y^0-x^0)[\phi^-(y),\phi^+(x)].\end{aligned} \tag{51}$$

Here, $\tau(t)$ is the time ordering step function $\tau(t)=1$ for $t>0$ and $\tau(t)=0$ for $t<0$. This indicates that the NPIA can be interpreted in the sense of a *total* time ordering acting with respect to the time argument of each field. On the other hand, the time ordering operator of IPTOPT just rearranges whole interactions. Now, let us consider ϕ^4-theory. We have

$$\begin{aligned}&(\phi\star\phi\star\phi\star\phi)(z) \\ &= \int\prod_{i=1}^{3}\left(\mathrm{d}^4 s_{\mathrm{i}}\frac{\mathrm{d}^4 l_i}{(2\pi)^4}\mathrm{e}^{\mathrm{i}l_i s_i}\right)\phi\left(z-\frac{1}{2}\tilde{l}_1\right)\phi\left(z+s_1-\frac{1}{2}\tilde{l}_2\right) \\ &\quad\times\phi\left(z+s_1+s_2-\frac{1}{2}\tilde{l}_3\right)\phi(z+s_1+s_2+s_3),\end{aligned} \tag{52}$$

which clearly expresses the non-locality. The two-point function at first order in λ can then be written a bit sloppy as

$$\begin{aligned}&G(x,y) \\ &= \frac{g}{4!}\int\mathrm{d}^4 z\,\langle 0\Big|T\big(\phi(x)\phi(y)\big(\phi\star\phi\star\phi\star\phi\big)(z)\big)\Big|0\rangle_{(0)}.\end{aligned} \tag{53}$$

[2] Clearly, this would not be equivalent to our IPTOPT approach. It would mean that we started with $H_{\mathrm{S}} = H[\phi_{\mathrm{H}},\pi_{\mathrm{H}};t=0]$.

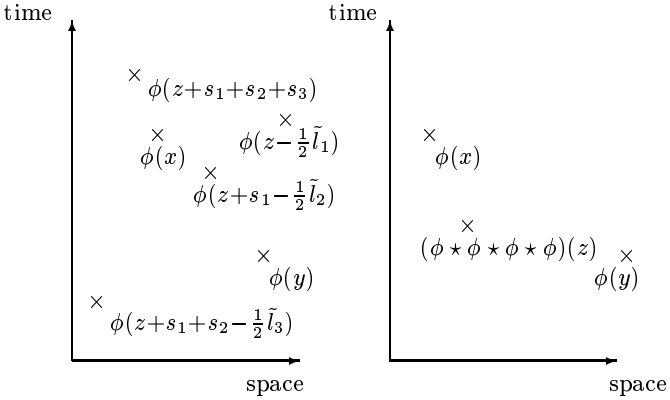

Fig. 1. Example of field arrangements

Let us discuss the total and the IPTOPT time ordering for one particular geometrical situation with respect to (53); see Fig. 1. The arrangement of fields for the left figure corresponds to the following non-vanishing contribution to the total time ordering of (53):

$$\begin{aligned}&G(x,y) \\ &= \int\mathrm{d}^4 z\int\prod_{i=1}^{3}\left(\mathrm{d}^4 s_i\frac{\mathrm{d}^4 l_i}{(2\pi)^4}\,e^{\mathrm{i}l_i s_i}\right) \\ &\quad\times\tau\left(s_1^0+s_2^0+s_3^0+\frac{1}{2}\tilde{l}_1^0\right)\tau\left(z^0-\frac{1}{2}\tilde{l}_1^0-x^0\right) \\ &\quad\times\tau\left(x^0-z^0-s_1^0+\frac{1}{2}\tilde{l}_2^0\right) \\ &\quad\times\tau\left(z^0+s_1^0-\frac{1}{2}\tilde{l}_2^0-y^0\right)\tau\left(y^0-z^0-s_1^0-s_2^0+\frac{1}{2}\tilde{l}_3^0\right) \\ &\quad\times\langle 0\Big|\phi(z+s_1+s_2+s_3)\phi\left(z-\frac{1}{2}\tilde{l}_1\right)\phi(x) \\ &\quad\times\phi\left(z+s_1-\frac{1}{2}\tilde{l}_2\right)\phi(y)\phi\left(z+s_1+s_2-\frac{1}{2}\tilde{l}_3\right)\Big|0\rangle_{(0)}.\end{aligned} \tag{54}$$

We find that there are $6!=720$ different contributions to (53) when interpreting the time ordering in the Gell-Mann–Low formula as a total time ordering of all field arguments, as one would expect. This kind of time ordering guarantees that only causal processes contribute to the S-matrix.

In contrast to this *total* time ordering, we now have *interaction point* time ordering (right figure), which is defined with respect to the *interaction point*:

$$\begin{aligned}&G'(x,y) \\ &= \int\mathrm{d}^4 z\int\prod_{i=1}^{3}\left(\mathrm{d}^4 s_i\frac{\mathrm{d}^4 l_i}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}l_i s_i}\right)\tau(x^0-z^0)\tau(z^0-y^0) \\ &\quad\times\langle 0\Big|\phi(x)\phi\left(z-\frac{1}{2}\tilde{l}_1\right)\phi\left(z+s_1-\frac{1}{2}\tilde{l}_2\right) \\ &\quad\times\phi\left(z+s_1+s_2-\frac{1}{2}\tilde{l}_3\right) \\ &\quad\times\phi(z+s_1+s_2+s_3)\phi(y)\Big|0\rangle_{(0)}.\end{aligned} \tag{55}$$

There are now only $3! = 6$ different contributions of this type. For most contributions some of the fields are now at the "wrong" place with respect to the total time order. Thus the non-commutative version (55) of the Gell-Mann–Low formula violates causality but preserves unitarity (as we want to stress once more). After all, contributions to the Dyson series are precisely ordered only with respect to the time stamp of the interaction Hamiltonians.

5 Gauge field theory

In this section, we will compute the non-commutative Hamiltonian for pure gauge theory, with and without ghosts. For simplicity, we restrict ourselves to the case of $U(1)$ gauge theory. We also do not employ Seiberg-Witten maps, but only replace the pointwise product of fields with the $*$-product (3).

5.1 Gauge fixed Lagrangian

The free part of the pure $U(1)$ gauge field Lagrangian on commutative space reads

$$\mathcal{L}_0 = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu)(\partial^\nu A_\nu), \tag{56}$$

where we have defined

$$f_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{57}$$

The free field equation reads

$$\Box A_\mu - \left(1 - \frac{1}{\alpha}\right) \partial_\mu(\partial^\nu A_\nu) = 0. \tag{58}$$

For the free field momenta we find

$$\Pi^i = f^{i0} = +f_{0i}, \quad \Pi^0 = -\frac{1}{\alpha}(\partial^\mu A_\mu). \tag{59}$$

Thus we define the *non-commutative* Hamiltonian

$$\begin{aligned}
H_0^* &= \int \mathrm{d}^3x\, \mathcal{H}_0^* \equiv \int \mathrm{d}^3x \left(\frac{1}{2}\{\dot{A}_\mu, \Pi^\mu\}_\star - \mathcal{L}_0^\star \right) \\
&= \int \mathrm{d}^3x \frac{1}{2}\{\dot{A}_i, f_{0i}\}_\star + \underbrace{\frac{1}{2}\partial^\mu A^\nu \star f_{\mu\nu}}_{-\frac{1}{4}\{\dot{A}_i, f_{0i}\}_\star - \frac{1}{4}\{\partial_i A_0, f_{i0}\}_\star + \frac{1}{2}\partial_i A_j \star f_{ij}} \\
&\quad - \frac{1}{2\alpha}\{\dot{A}_0, \partial^\mu A_\mu\}_\star + \frac{1}{2\alpha}(\partial^\mu A_\mu) \star (\partial^\nu A_\nu) \\
&= \int \mathrm{d}^3x \frac{1}{4}\{(\dot{A}_i + \partial_i A_0), f_{0i}\}_\star + \underbrace{\frac{1}{2}\partial_i A_j \star f_{ij}}_{\frac{1}{2}(\partial_i A_j \star \partial_i A_j - \partial_i A_i \star \partial_j A_j)} \\
&\quad - \frac{1}{2\alpha}(\dot{A}_0) \star (\dot{A}_0) + \frac{1}{2\alpha}(\partial_i A_i) \star (\partial_j A_j) \\
&\quad + \dot{A}_0 \star \dot{A}_0 - \dot{A}_0 \star \dot{A}_0 \\
&= \frac{1}{2}\int \mathrm{d}^3x \Big(\dot{A}_i \star \dot{A}_i - \dot{A}_0 \star \dot{A}_0 + \partial_j A_i \star \partial_j A_i \\
&\quad - \partial_j A_0 \star \partial_j A_0 \\
&\quad + \left(\frac{\alpha - 1}{\alpha}\right)(\dot{A}_0 \star \dot{A}_0 - (\partial_i A_i) \star (\partial_j A_j)) \Big).
\end{aligned} \tag{60}$$

We check explicitly the time independence of the Hamiltonian,

$$\begin{aligned}
\dot{H}_0^* = \frac{1}{2}\int \mathrm{d}^3x \frac{1}{2} \Big\{ & \dot{A}_i, \ddot{A}_i - \boldsymbol{\nabla}^2 A_i \\
& + \left(\frac{\alpha - 1}{\alpha}\right)(-\partial_i \partial^\mu A_\mu + \underbrace{\partial_i \dot{A}_0}) \Big\}_\star \\
& - \frac{1}{2}\left\{ \dot{A}_0, \ddot{A}_0 - \boldsymbol{\nabla}^2 A_0 - \left(\frac{\alpha - 1}{\alpha}\right)\ddot{A}_0 \right\}_\star,
\end{aligned} \tag{61}$$

where $\boldsymbol{\nabla}^2 = \partial_j \partial_j$. Note that the use of the Moyal anticommutators is crucial! The first line (without the underbraced term) is zero due to the equation of motion for A_i. After partial integration of the underbraced term with respect to $\partial_i = -\partial^i$ it combines with the second line to the equation of motion for A_0. Thus we see that $\frac{\mathrm{d}}{\mathrm{d}t} H_0^* = 0$.

For quantisation we rewrite H_0^* in a convenient form:

$$\begin{aligned}
H_0^* = \frac{1}{2}\int \mathrm{d}^3x \Big(& -\partial_\mu A^\nu \star \partial_\mu A_\nu \\
& + \left(\frac{\alpha - 1}{2\alpha}\right)\{\partial^\mu A_\mu, \partial_\nu A_\nu\}_\star \Big).
\end{aligned} \tag{62}$$

We make the ansatz

$$A_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \Big(a_\mu^+(\boldsymbol{k}) \mathrm{e}^{+\mathrm{i}kx} + a_\mu^-(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx} \Big), \tag{63}$$

where $k_0 = \omega_k > 0$ is a (not necessarily specified) function of $|\boldsymbol{k}|$. Inserting this into the expression for H_0^* we find

$$\begin{aligned}
H_0^* = \frac{1}{2}\int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \int \frac{\mathrm{d}^3q}{\sqrt{2\omega_q}} & \\
\times \Big(& \mathrm{e}^{\mathrm{i}t(\omega_k + \omega_q) - \frac{\mathrm{i}}{2}\theta^{\mu\nu} k_\mu q_\nu} \delta^3(\boldsymbol{k} + \boldsymbol{q}) \\
\times \Big(& k_\mu q_\mu a^{\nu +}(\boldsymbol{k}) a_\nu^+(\boldsymbol{q}) \\
& + \left(\frac{\alpha - 1}{2\alpha}\right) \\
& \times (-k^\mu q_\nu a_\mu^+(\boldsymbol{k}) a_\nu^+(\boldsymbol{q}) - k_\nu q^\mu a_\nu^+(\boldsymbol{k}) a_\mu^+(\boldsymbol{q})) \Big) \\
& + \mathrm{e}^{-\mathrm{i}t(\omega_k + \omega_q) - \frac{\mathrm{i}}{2}\theta^{\mu\nu} k_\mu q_\nu} \delta^3(\boldsymbol{k} + \boldsymbol{q}) \\
\times \Big(& k_\mu q_\mu a^{\nu -}(\boldsymbol{k}) a_\nu^-(\boldsymbol{q}) \\
& + \left(\frac{\alpha - 1}{2\alpha}\right) \\
& \times (-k^\mu q_\nu a_\mu^-(\boldsymbol{k}) a_\nu^-(\boldsymbol{q}) - k_\nu q^\mu a_\nu^-(\boldsymbol{k}) a_\mu^-(\boldsymbol{q})) \Big)
\end{aligned} \tag{64}$$

$$
\begin{aligned}
&+\mathrm{e}^{\mathrm{i}t(\omega_k-\omega_q)+\frac{\mathrm{i}}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta^3(\boldsymbol{k}-\boldsymbol{q})\\
&\times\Bigg(-k_\mu q_\mu a^{\nu+}(\boldsymbol{k})a^-_\nu(\boldsymbol{q})\\
&+\left(\frac{\alpha-1}{2\alpha}\right)\\
&\times(+k^\mu q_\nu a^+_\mu(\boldsymbol{k})a^-_\nu(\boldsymbol{q})+k^\mu q_\nu a^-_\nu(\boldsymbol{k})a^+_\mu(\boldsymbol{q}))\Bigg)\\
&+\mathrm{e}^{-\mathrm{i}t(\omega_k-\omega_q)+\frac{\mathrm{i}}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta^3(\boldsymbol{k}-\boldsymbol{q})\\
&\times\Bigg(-k_\mu q_\mu a^{\nu-}(\boldsymbol{k})a^+_\nu(\boldsymbol{q})\\
&+\left(\frac{\alpha-1}{2\alpha}\right)\\
&\times(+k^\mu q_\nu a^-_\mu(\boldsymbol{k})a^+_\nu(\boldsymbol{q})+k^\mu q_\nu a^+_\nu(\boldsymbol{k})a^-_\mu(\boldsymbol{q}))\Bigg)\Bigg)\Bigg).
\end{aligned}
$$

Using now the delta functions we find

$$
\begin{aligned}
H_0^* &= \frac{1}{2}\int\frac{\mathrm{d}^3k}{\sqrt{2\omega_k}}\int\frac{\mathrm{d}^3q}{\sqrt{2\omega_q}}\\
&\times\Bigg(\mathrm{e}^{\mathrm{i}t(\omega_k+\omega_q)-\frac{\mathrm{i}}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta^3(\boldsymbol{k}+\boldsymbol{q})\\
&\times\Bigg(\frac{1}{2}(k^2a^{\nu+}(\boldsymbol{k})a^+_\nu(\boldsymbol{q})+q^2a^+_\nu(\boldsymbol{k})a^{\nu+}(\boldsymbol{q}))\\
&+\left(\frac{\alpha-1}{2\alpha}\right)\\
&\times(-k^\mu k^\nu a^+_\mu(\boldsymbol{k})a^+_\nu(\boldsymbol{q})-q^\nu q^\mu a^+_\nu(\boldsymbol{k})a^+_\mu(\boldsymbol{q}))\Bigg)\\
&+\mathrm{e}^{-\mathrm{i}t(\omega_k+\omega_q)-\frac{\mathrm{i}}{2}\theta^{\mu\nu}k_\mu q_\nu}\delta^3(\boldsymbol{k}+\boldsymbol{q})\\
&\times\Bigg(\frac{1}{2}\Bigg(k^2a^{\nu-}(\boldsymbol{k})a^-_\nu(\boldsymbol{q})\\
&+q^2a^-_\nu(\boldsymbol{k})a^{\nu-}(\boldsymbol{q})\\
&+\left(\frac{\alpha-1}{2\alpha}\right)\\
&\times(-k^\mu k^\nu a^-_\mu(\boldsymbol{k})a^-_\nu(\boldsymbol{q})-q^\nu q^\mu a^-_\nu(\boldsymbol{k})a^-_\mu(\boldsymbol{q}))\Bigg)\\
&+\delta^3(\boldsymbol{k}-\boldsymbol{q})\Bigg(-k_\mu k_\mu a^{\nu+}(\boldsymbol{k})a^-_\nu(\boldsymbol{k})\\
&+\left(\frac{\alpha-1}{2\alpha}\right)\\
&\times(+k^\mu k_\nu a^+_\mu(\boldsymbol{k})a^-_\nu(\boldsymbol{k})+k^\mu k_\nu a^-_\nu(\boldsymbol{k})a^+_\mu(\boldsymbol{k}))\Bigg)\\
&+\delta^3(\boldsymbol{k}-\boldsymbol{q})\Bigg(-k_\mu k_\mu a^{\nu-}(\boldsymbol{k})a^+_\nu(\boldsymbol{k})\\
&+\left(\frac{\alpha-1}{2\alpha}\right)\\
&\times(+k^\mu k_\nu a^-_\mu(\boldsymbol{k})a^+_\nu(\boldsymbol{k})+k^\mu k_\nu a^+_\nu(\boldsymbol{k})a^-_\mu(\boldsymbol{k}))\Bigg)\Bigg)\Bigg).
\end{aligned}
$$

With the equation of motion (58) expressed in terms of $a^\pm_\mu$,

$$
k^2a^\pm_\mu(\boldsymbol{k})-\left(\frac{\alpha-1}{\alpha}\right)k_\mu(k^\nu a^\pm_\nu(\boldsymbol{k}))=0, \tag{64}
$$

the terms with the non zero exponentials vanish. The remaining terms simplify considerably with the help of the equation of motion. So we get

$$
\begin{aligned}
H_0^* &= \frac{1}{2}\int\frac{\mathrm{d}^3k}{2\omega_k}\big(-k_\mu k_\mu a^{\nu+}(\boldsymbol{k})a^-_\nu(\boldsymbol{k})+k^2a^+_\nu(\boldsymbol{k})a^-_\nu(\boldsymbol{k})\\
&\quad -k_\mu k_\mu a^{\nu-}(\boldsymbol{k})a^+_\nu(\boldsymbol{k})+k^2a^-_\nu(\boldsymbol{k})a^+_\nu(\boldsymbol{k})\big) \qquad (65)\\
&= \int\frac{\mathrm{d}^3k}{2\omega_k}\big(-\boldsymbol{k}^2(a^+_0a^-_0+a^-_0a^+_0)+\omega_k^2(a^+_ia^-_i+a^-_ia^+_i)\big)\\
&= H_0.
\end{aligned}
$$

Quantisation can now be performed in the usual way by imposing appropriate commutator relations (e.g. for $\alpha=1$, Feynman gauge)

$$
[a^-_\rho(\boldsymbol{k}),a^+_\mu(\boldsymbol{k}')]=-g_{\rho\mu}\delta^3(\boldsymbol{k}-\boldsymbol{k}'). \tag{66}
$$

5.2 BRST-symmetry

The free part of the BRST-expanded Lagrangian on a non-commutative space reads

$$
\begin{aligned}
L_0 = \int\mathrm{d}^3x\Bigg(&-\frac{1}{4}f_{\mu\nu}f^{\mu\nu}+B\left(\partial^\mu A_\mu+\frac{\alpha}{2}B\right) \qquad (67)\\
&+\partial^\mu\bar{c}\partial_\mu c\Bigg).
\end{aligned}
$$

The equations of motion read

$$
\begin{aligned}
\frac{\partial\mathcal{L}_0}{\partial A^\mu} &= \Box A_\mu-\partial_\mu(\partial^\nu A_\nu)-\partial_\mu B=0,\\
\frac{\partial\mathcal{L}_0}{\partial B} &= \partial^\mu A_\mu+\alpha B=0. \qquad (68)
\end{aligned}
$$

We postpone the treatment of the ghost sector, which in the free theory decouples from the gauge field sector anyway. In order to construct the Hamiltonian we have

$$
\begin{aligned}
\frac{\partial\mathcal{L}_0}{\partial\dot{A}^i} &=: \Pi_i=f_{i0},\\
\frac{\partial\mathcal{L}_0}{\partial\dot{A}^0} &=: \Pi_0=B, \quad \frac{\partial\mathcal{L}_0}{\partial\dot{B}}=:\Pi_B=0. \qquad (69)
\end{aligned}
$$

The latter two equations are primary constraints. Since their Poisson bracket is not weakly zero,

$$
\begin{aligned}
\{\phi_1(\boldsymbol{x}),\phi_2(\boldsymbol{x}')\}_{\mathrm{PB}} &= \{\Pi_0(\boldsymbol{x})-B(\boldsymbol{x}),\Pi_B(\boldsymbol{x}')\}_{\mathrm{PB}}\\
&= -\delta^3(\boldsymbol{x}-\boldsymbol{x}'), \qquad (70)
\end{aligned}
$$

they are second class constraints.

Now, in order to write down the corresponding *non-commutative* Hamiltonian, we firstly define the symmetrized $\star_s$-product,

$$A \star_s B = \frac{1}{2}(A \star B \pm B \star A), \tag{71}$$

where the sign is positive for usual fields and negative for Grassmann valued fields. Again, the use of this $\star_s$-product is crucial.

The total non-commutative Hamiltonian [21] thus reads (with use of $\dot{A}_i \star_s \Pi^i = (\partial_i A_0 - \Pi_i) \star_s \Pi^i$ and partial integration)

$$\begin{aligned} H_T^* &= \int \mathrm{d}^3x \Big(\dot{A}_\mu \star_s \Pi^\mu + \dot{B} \star_s \Pi_B - \mathcal{L}_0 + \lambda_1' \star_s \phi_1 \\ &\quad + \lambda_2' \star_s \phi_2 \Big) \\ &= \int \mathrm{d}^3x \left(\underbrace{(\lambda_1' + \dot{A}_0)}_{\lambda_1} \star_s (\Pi^0 - B) + \underbrace{(\lambda_2' + \dot{B})}_{\lambda_2} \star_s \Pi_B \right. \\ &\quad - A_0 \star_s \partial_i \Pi^i - B \star_s \partial_i A^i - \frac{\alpha}{2} B \star B \\ &\quad \left. + \frac{1}{2} \Pi^i \star \Pi^i + \frac{1}{4} f^{ij} \star f^{ij} \right), \end{aligned} \tag{72}$$

where ϕ_1 and ϕ_2 denote the constraints defined in (70), λ_1' and λ_2' are Lagrange multiplier. Since the constraints should be preserved in time, we find conditions for λ_i,

$$\begin{aligned} \{H_T^*, \phi_1\}_{\mathrm{PB}} &= \lambda_2 - \partial_i \Pi^i = 0, \\ \{H_T^*, \phi_2\}_{\mathrm{PB}} &= -\lambda_1 - \partial_i A^i - \alpha B = 0. \end{aligned} \tag{73}$$

According to Dirac [21], for quantisation the second class constraints are imposed as strong operator equations. This is only possible after elimination of the unphysical degrees of freedom corresponding to the second class constraints. Clearly, these degrees of freedom are simply $B,\ \Pi_B$.

So, with $\Pi_B = 0$ and $B = \Pi^0$ we get the quantisable non-commutative Hamiltonian

$$\begin{aligned} H'^* = \int \mathrm{d}^3x \Big(& -A_0 \star_s \partial_i \Pi^i - \Pi^0 \star_s \partial_i A^i \\ & - \frac{\alpha}{2} \Pi^0 \star \Pi^0 + \frac{1}{2} \Pi^i \star \Pi^i + \frac{1}{4} f^{ij} \star f^{ij} \Big). \end{aligned} \tag{74}$$

With use of the Hamiltonian equations of motion for the fields,

$$\begin{aligned} \dot{A}_0 &= \frac{\delta H'^*}{\delta \Pi^0} = -\partial_i A^i - \alpha \Pi^0, \\ \dot{A}_i &= \frac{\delta H'^*}{\delta \Pi^i} = \partial_i A^0 - \Pi_i, \end{aligned} \tag{75}$$

we may express the field momenta by the fields and their time derivative. Inserting this yields exactly the Hamiltonian (60) we have found for the gauge fixed theory:

$$\begin{aligned} H'^* = \int \mathrm{d}^3x \Big(& (-A^0 \star_s \partial^i \dot{A}^i + A^0 \star_s \partial^i \partial^i A^0) \\ & + \left(\frac{1}{\alpha} \partial_i A^i \star \partial_j A^j + \frac{1}{\alpha} \dot{A}^0 \star_s \partial_j A^j \right) \\ & + \left(-\frac{1}{2\alpha} \partial_i A^i \star \partial_j A^j - \frac{1}{2\alpha} \dot{A}^0 \star \dot{A}^0 - \frac{2}{2\alpha} \partial_i A^i \star_s \dot{A}^0 \right) \\ & + \left(\frac{1}{2} \dot{A}^i \star \dot{A}^i + \frac{1}{2} \partial^i A^0 \star \partial^i A^0 - \dot{A}^i \star_s \partial^i A^0 \right) \\ & + \left(\frac{1}{2} \partial^i A^j \star \partial^i A^j - \frac{1}{2} \partial^i A^j \star \partial^j A^i \right) \Big) \\ = \frac{1}{2} \int \mathrm{d}^3x \Big(& \dot{A}_i \star \dot{A}_i - \dot{A}_0 \star \dot{A}_0 + \partial_j A_i \star \partial_j A_i \\ & - \partial_j A_0 \star \partial_j A_0 \\ & + \left(\frac{\alpha - 1}{\alpha} \right) (\dot{A}_0 \star \dot{A}_0 - (\partial_i A_i) \star (\partial_j A_j)) \Big). \end{aligned} \tag{76}$$

Note that the elimination of the B field does not spoil our considerations with respect to the construction of perturbation theory, since the B field has no interaction vertex.

Now for $c,\ \bar{c}$ the situation is very simple,

$$L_{\phi\Pi} = \int \mathrm{d}^3x \mathcal{L}_{\phi\Pi} = \int \mathrm{d}^3x \partial^\mu \bar{c} \star_s \partial_\mu\ c. \tag{77}$$

The equations of motion and the momenta are

$$\begin{aligned} \frac{\partial \mathcal{L}_{\phi\Pi}}{\partial \bar{c}} &= -\Box c = 0, \quad \frac{\partial \mathcal{L}_{\phi\Pi}}{\partial c} = \Box \bar{c} = 0, \\ \frac{\partial \mathcal{L}_{\phi\Pi}}{\partial \dot{\bar{c}}} &= \Pi_{\bar{c}} = \dot{c}, \qquad \frac{\partial \mathcal{L}_{\phi\Pi}}{\partial \dot{c}} = \Pi_c = -\dot{\bar{c}}. \end{aligned} \tag{78}$$

There are no constraints. For the non-commutative Hamiltonian we have

$$\begin{aligned} H_{\phi\Pi}^* &= \int \mathrm{d}^3x (\dot{\bar{c}} \star_s \Pi_{\bar{c}} + \dot{c} \star_s \Pi_c + \Pi_c \star_s \Pi_{\bar{c}} + \partial_i \bar{c} \star_s \partial_i\ c) \\ &= \int \mathrm{d}^3x (\dot{\bar{c}} \star_s \dot{c} - \dot{c} \star_s \dot{\bar{c}} - \dot{\bar{c}} \star_s \dot{c} + \partial_i \bar{c} \star_s \partial_i c) \\ &= \int \mathrm{d}^3x (\dot{\bar{c}} \star_s \dot{c} + \partial_i \bar{c} \star_s \partial_i\ c). \end{aligned} \tag{79}$$

We check time independence of $H_{\phi\Pi}^*$,

$$\begin{aligned} \dot{H}_{\phi\Pi}^* &= \int \mathrm{d}^3x \Big((\ddot{\bar{c}} - \partial_i \partial_i \bar{c}) \star_s \dot{c} + \dot{\bar{c}} \star_s (\ddot{c} - \partial_i \partial_i c) \Big) \\ &= 0. \end{aligned} \tag{80}$$

Using the following ansatz for $c, \bar{c}$:

$$\begin{aligned} c(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \Big(c^+(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} + c^-(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx} \Big), \\ \bar{c}(x) &= \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3k}{\sqrt{2\omega_k}} \Big(\bar{c}^+(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} - \bar{c}^-(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx} \Big) \end{aligned} \tag{81}$$

(note that $\bar{c}(x)$ is here imaginary) and the Poisson bracket for Grassmann fields, $\{c(\boldsymbol{x}), \Pi_c(\boldsymbol{x}')\} = \{\bar{c}(\boldsymbol{x}), \Pi_{\bar{c}}(\boldsymbol{x}')\} = -\delta^3(\boldsymbol{x} - \boldsymbol{x}')$, we find

$$\{\bar{c}^+(\boldsymbol{k}), c^-(\boldsymbol{k}'\} = \{\bar{c}^-(\boldsymbol{k}), c^+(\boldsymbol{k}')\} = -\mathrm{i}\delta^3(\boldsymbol{k} - \boldsymbol{k}'). \tag{82}$$

For $H^*_{\phi\Pi}$ we find with the help of the equations of motion (with $k_0 = \omega_k = \sqrt{\boldsymbol{k}^2}$, so that $k_\mu k_\mu = 2\omega_k^2$)

$$\begin{aligned} H^*_{\phi\Pi} &= \int \mathrm{d}^3x \partial_\mu \bar{c}(x) \star_s \partial_\mu c(x) \\ &= \int \frac{\mathrm{d}^3x}{(2\pi)^3} \int \frac{\mathrm{d}^3k\, \mathrm{d}^3k'}{2\sqrt{\omega_k \omega_{k'}}} \\ &\quad \times \big((+\mathrm{i}k_\mu \bar{c}^+(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}kx} + \mathrm{i}k_\mu \bar{c}^-(\boldsymbol{k}) \mathrm{e}^{-\mathrm{i}kx}) \\ &\qquad \times \star_s (+\mathrm{i}k'_\mu c^+(\boldsymbol{k}') \mathrm{e}^{\mathrm{i}k'x} - \mathrm{i}k'_\mu c^-(\boldsymbol{k}') \mathrm{e}^{-\mathrm{i}k'x}) \big) \\ &= \int \mathrm{d}^3k \omega_k \left(\bar{c}^+(\boldsymbol{k}) c^-(\boldsymbol{k}) + c^+(\boldsymbol{k}) \bar{c}^-(\boldsymbol{k}) \right) \\ &= H_{\phi\Pi}. \end{aligned} \tag{83}$$

We find that non-commutativity does not spoil the free theory. Quantisation is done by the replacement of the Poisson brackets by commutators,

$$\{\,,\,\}_{\mathrm{PB}} \Rightarrow -\mathrm{i}[\,,\,], \tag{84}$$

which again leads to the well known commutator relations between annihilation and creation operators of fields.

6 Conclusion

We have constructed perturbation theory on a non-commutative space from the beginning. We have discussed a general non-local deformation of the free Hamiltonian. For some of the considered deformations, such as the canonical one [1], the free Hamiltonian is unaffected by the deformation. In the canonical case, this is even true for gauge field theory as shown in the previous section. This implies that IPTOPT as developed in [1,6,8,9] is consistent. However, the application of these techniques to Gaussian non-localities [16,17] seems to be somewhat artificial since here we could show that the introduction of the non-localities into the free Hamiltonian would alter it significantly when assuming the usual free field equations. Thus, either one has to leave the free Hamiltonian untouched [17] or one has to develop an appropriate free theory.

Furthermore, the discrepancies between IPTOPT and the NPIA present for non-localities in time have been discussed in some detail. The main reason is the problem of passing from non-local products given in the Dirac picture to the Heisenberg picture. One has to alter these products in order for them to be consistent. But usually one compares situations where one deals with the same product in both pictures, which cannot give the same as soon as non-localities in time are involved. We also want to point out that the use of L_{I} instead of H_{I} in combination with IPTOPT is not justified by path integrals as soon as non-localities involve time.

The main motivation for studying IPTOPT is its unitarity [7] and the well behaving UV/IR mixing [10]. The disadvantages are the violation of causality and the higher complexity of the Feynman rules. The situation is vice versa for the NPIA, where the implied time ordering respects causality. The violation of unitarity in NPIA is a severe problem, whereas it seems to be absent in IPTOPT.

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